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A note on the equivalences between the averages and the K-functionals related to the Laplacian

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Abstract

For $\mathbb{R}^d$ or $\mathbb{T}^d$, a strong converse inequality of type A (in the terminology of Ditzian and Ivanov (J. Anal. Math. 61 (1993) 61)) is obtained for the high order averages on balls and the K-functionals generated by the high order Laplacian, which answers a problem raised by Ditzian and Runovskii (J. Approx. Theory 97 (1999) 113).

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1. Introduction and main result

Given a function $f \in L(\mathbb{R}^d)$, its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}^d.$$

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For a positive integer ℓ, the ℓth order Laplacian $\triangle^\ell$ is defined, in a distributional sense, by

$$(\triangle^\ell f)^\wedge(\xi) = (-1)^\ell |\xi|^{2\ell} \widehat{f}(\xi).$$

Associated with the operator $\triangle^\ell$, there is a *K*-functional

$$K_{\triangle,\ell}(f, t^{2\ell})_p := \inf\{\|f-g\|_p + t^{2\ell}\|\triangle^\ell g\|_p :\ g,\ \triangle^\ell g \in L^p(\mathbb{R}^d)\}, \tag{1}$$

where $t > 0$, $1 \leqslant p \leqslant \infty$ and $\|\cdot\|_p$ denotes the usual L^p-norm on $\mathbb{R}^d$.

Let V_d denote the volume of the unit ball of $\mathbb{R}^d$. For $t > 0$ and a locally integrable function *f*, we define the average $B_t(f)$ by

$$B_t(f)(x) = \frac{1}{t^d V_d} \int_{\{u\in\mathbb{R}^d:\ |u|\leqslant t\}} f(x+u)\, du$$

and the ℓth order average $B_{\ell,t}(f)$ (for a given positive integer ℓ) by

$$B_{\ell,t}(f)(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt}(f)(x). \tag{2}$$

We remark that for $\ell > 1$ the operator $B_{\ell,t}$ was first introduced by Ditzian and Runovskii in [DR, p. 117, (2.6)].

For more background information we refer to [Di1,Di2,DR,Di-Iv,To].

Our main goal in this paper is to prove the following strong converse inequality of type A (in the terminology of [Di-Iv]), which was conjectured in [DR, p. 138].

Theorem 1. *Let* $\ell \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ *and* $f \in L^p(\mathbb{R}^d)$. *Then*

$$\|f - B_{\ell,t}(f)\|_p \approx K_{\triangle,\ell}(f, t^{2\ell})_p,$$

where $t > 0$ *and*

$$A(f,t) \approx B(f,t)$$

means that there is a $C > 0$, *independent of f and t, such that*

$$C^{-1} A(f,t) \leqslant B(f,t) \leqslant C A(f,t).$$

Theorem 1 for $\ell = 1$ was proved in [DR, p. 133, Theorem 6.1] and for $d = 1$, ℓ small, as it was indicated in [DR, p. 138], can be obtained by following the technique developed in [Di-Iv]. For $\ell \geqslant 2$ and $d \geqslant 2$, the following strong converse inequality of type B (in the terminology of [Di-Iv]) was obtained in [DR, p. 127, Theorem 4.8 and p. 131, Theorem 5.7]:

$$K_{\triangle,\ell}(f, t^{2\ell})_p \approx \|f - B_{\ell,t}(f)\|_p + \|f - B_{\ell,t\rho}(f)\|_p,\ 1 \leqslant p \leqslant \infty \tag{3}$$

for some $\rho > 1$. The proof of our Theorem 1 will be based on this equivalence.

We remark that with a slight modification of the proof below a similar result for the periodic case can also be obtained.

2. Basic lemmas

The following lemma can be easily obtained by a straightforward computation.

Lemma 1. *Let* $\chi_{B(0,1)}(x)$ *denote the characteristic function of the unit ball*

$$B(0,1) := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \ x_1^2 + \cdots + x_d^2 \leqslant 1\},$$

V_d *denote the volume of* $B(0,1)$ *and let* $I(x) = \frac{1}{V_d}\chi_{B(0,1)}(x)$. *Then*

$$\widehat{I}(x) = \gamma_d \int_0^1 \cos(u|x|)(1-u^2)^{\frac{d-1}{2}}\,du \tag{4}$$

with

$$\gamma_d = \Big(\int_0^1 (1-u^2)^{\frac{d-1}{2}}\,du\Big)^{-1}. \tag{5}$$

Lemma 2. *Let* $B_{\ell,t}$ *be defined by* (2) *and* $I(x)$ *the same as in Lemma* 1. *Then for* $f \in L(\mathbb{R}^d)$,

$$\widehat{B_{\ell,t}(f)}(x) = m_\ell(t|x|)\widehat{f}(x), \tag{6}$$

where

$$m_\ell(|x|) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} \widehat{I}(jx) \tag{7}$$

$$= 1 - A_\ell(|x|), \tag{8}$$

$$A_\ell(|x|) = \gamma_d \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{d-1}{2}} (\sin \frac{u|x|}{2})^{2\ell}\,du \tag{9}$$

and γ_d *is given by* (5).

Proof. For $t > 0$, we write

$$I_t(x) = \frac{1}{t^d} I(\frac{x}{t}).$$

Then from definition (2), it follows that

$$B_{\ell,t}(f)(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} (f * I_{jt})(x),$$

which implies (6) and (7). Substituting (4) into (7) yields

$$m_\ell(|x|) = \frac{-2\gamma_d}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} \int_0^1 \cos(ju|x|)(1-u^2)^{\frac{d-1}{2}}\,du, \tag{10}$$

which, together with the following identity

$$(\sin \frac{x}{2})^{2\ell} = \frac{\binom{2\ell}{\ell}}{4^\ell} + \frac{2}{4^\ell} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \cos jx,$$

gives (8) and (9). This completes the proof. □

Lemma 3. *Let $m_\ell(u)$ be the same as in Lemma 2. Then for $j \in \mathbb{Z}_+$ and $u \geqslant 0$,*

$$\left|\left(\frac{d}{du}\right)^j m_\ell(u)\right| \leqslant C_{\ell,j} \left(\frac{1}{u+1}\right)^{\frac{d+1}{2}},$$

where $C_{\ell,j} > 0$ is independent of u.

Proof. By identity (10), it suffices to show that for $j \in \mathbb{Z}_+$ and $u \geqslant 0$,

$$\left|\left(\frac{d}{du}\right)^j \int_0^1 \cos(uv)(1-v^2)^{\frac{d-1}{2}}\, dv\right| \leqslant C_j \left(\frac{1}{u+1}\right)^{\frac{d+1}{2}}. \tag{11}$$

We use formula (4.7.5) of [An-As-R, p. 204] to obtain that

$$\int_0^1 \cos(uv)(1-v^2)^{\frac{d-1}{2}}\, dv = 2^{\frac{d-2}{2}} \sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}\right) \frac{J_{\frac{d}{2}}(u)}{u^{\frac{d}{2}}}, \tag{12}$$

where $J_\alpha(u)$ denotes the Bessel function of the first kind of order α. Now (11) is a consequence of (12) and the following well-known estimates on Bessel functions:

$$\frac{d}{du} u^{-\alpha} J_\alpha(u) = -u^{-\alpha} J_{\alpha+1}(u), \quad \text{[An-As-R, (4.6.2), p. 202]},$$

$$J_\alpha(u) = O\left(\frac{1}{(u+1)^{\frac{1}{2}}}\right) \quad \text{for } u \geqslant 0 \quad \text{[An-As-R, (4.8.5), p. 209]},$$

$$J_\alpha(u) = O(u^\alpha) \text{ as } u \to 0 \quad \text{[An-As-R, (4.7.6), p. 218]}.$$

This concludes the proof. □

Lemma 4. *Suppose that a is a C^∞-function defined on $[0, \infty)$ with the property that for $u \geqslant 0$ and $0 \leqslant j \leqslant d+1$,*

$$\left|\left(\frac{d}{du}\right)^j a(u)\right| \leqslant C(a) \left(\frac{1}{1+u}\right)^{d+1}. \tag{13}$$

For $t > 0$, define the operator T_t, in a distributional sense, by

$$(T_t(f))^\wedge(\xi) = a(t|\xi|)\hat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

Then for $1 \leqslant p \leqslant \infty$ and $f \in L^p(\mathbb{R}^d)$,

$$\sup_{t>0} \|T_t(f)\|_p \leqslant C_{p,a} \|f\|_p.$$

This lemma is well known (see [St]), but for the sake of completeness, we give its proof here.

Proof. Let

$$K(x) = \int_{\mathbb{R}^d} e^{ix\xi} a(|\xi|)\, d\xi. \tag{14}$$

Since

$$T_t(f)(x) = f * K_t(x),$$

with

$$K_t(x) = \frac{1}{t^d} K(\frac{x}{t}),$$

it is sufficient to prove

$$\|K\|_{L^1(\mathbb{R}^d)} < \infty. \tag{15}$$

By (14), we get for $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d$,

$$(-x)^\gamma K(x) = \int_{\mathbb{R}^d} e^{ix\xi} \Big(\frac{\partial}{\partial \xi}\Big)^\gamma (a(|\xi|))\, d\xi,$$

which, by (13), implies

$$|x^\gamma K(x)| \leqslant C \int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|)^{d+1}} < \infty,$$

with $|\gamma| = \gamma_1 + \cdots + \gamma_d \leqslant d+1$. Now taking the supremum over all γ with $|\gamma| = d+1$ yields

$$|K(x)| \leqslant \frac{C}{|x|^{d+1}},$$

which, together with the fact that $K \in C(\mathbb{R}^d)$, implies (15) and so completes the proof. □

3. Proof of Theorem 1

The upper estimate

$$\|f - B_{\ell,t}(f)\|_p \leqslant C_{\ell,p} K_{\triangle,\ell}(f, t^{2\ell})_p$$

follows directly from (3), which, as indicated in Section 1, was proved in [DR]. Hence it remains to prove the lower estimate

$$\|f - B_{\ell,t}(f)\|_p \geqslant C_{\ell,p} K_{\triangle,\ell}(f, t^{2\ell})_p.$$

Lemma 3 implies that there is a number $\mu = \mu(\ell, d) > 1$ such that for $u > \mu$,

$$|m_\ell(u)| \leqslant \frac{1}{2}. \tag{16}$$

We will keep this special number μ throughout the proof.

Let η be a C^∞-function on $[0, \infty)$ with the properties that $\eta(x) = 0$ for $x > 2$, $\eta(x) = 1$ for $0 \leqslant x \leqslant 1$, and $0 \leqslant \eta(x) \leqslant 1$ for all $x \in [0, \infty)$. For $t > 0$, we define the operator V_t by

$$(V_t(f))^\wedge(\xi) = \eta(t|\xi|)\widehat{f}(\xi), \tag{17}$$

where $f \in L^p(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$.

According to definition (1), the estimates

$$\|f - V_{t/2\mu}(f)\|_p \leqslant C_{\ell,p}\|f - B_{\ell,t}(f)\|_p \tag{18}$$

and

$$t^{2\ell}\|\triangle^\ell V_{t/2\mu}(f)\|_p \leqslant C_{\ell,p}\|f - B_{\ell,t}(f)\|_p \tag{19}$$

will prove

$$K_{\triangle,\ell}(f, t^{2\ell})_p \leqslant \|f - V_{t/2\mu}(f)\|_p + t^{2\ell}\|\triangle^\ell V_{t/2\mu}(f)\|_p \leqslant C_{\ell,p}\|f - B_{\ell,t}(f)\|_p$$

and so complete the proof of Theorem 1. Thus, it has remained to prove (18) and (19).

Let

$$\phi(u) = \left(1 - \eta(\frac{u}{2\mu})\right)\frac{(m_\ell(u))^3}{1 - m_\ell(u)} \tag{20}$$

and

$$\psi(u) = \frac{u^{2\ell}\eta(\frac{u}{2\mu})}{A_\ell(u)}, \tag{21}$$

with $A_\ell(u)$ and $m_\ell(u)$ the same as in Lemma 2. For $t > 0$, we define two operators Φ_t and Ψ_t as follows:

$$\begin{aligned}\left(\Phi_t(f)\right)^\wedge(\xi) &:= \phi(t|\xi|)\hat{f}(\xi),\\ \left(\Psi_t(f)\right)^\wedge(\xi) &:= \psi(t|\xi|)\hat{f}(\xi).\end{aligned} \tag{22}$$

It follows from (16), (20) and Lemma 3 that for $u \geqslant 0$ and $0 \leqslant j \leqslant d+1$,

$$|\phi^{(j)}(u)| \leqslant C_{\ell,d}\left(\frac{1}{u+1}\right)^{\frac{3(d+1)}{2}}. \tag{23}$$

On the other hand, by (9) and a straightforward computation, we obtain that for $u \geqslant \frac{\pi}{2}$

$$A_\ell(u) \geqslant C_{\ell,d}\int_0^{\frac{2}{3}}(\sin\frac{uv}{2})^{2\ell}\,dv \geqslant C'_{\ell,d} > 0 \tag{24}$$

and for $0 < u < \frac{\pi}{2}$

$$\frac{A_\ell(u)}{u^{2\ell}} \geqslant C_{\ell,d} \frac{1}{u^{2\ell}} \int_0^1 (1-v^2)^{\frac{d-1}{2}} (uv)^{2\ell}\, dv \geqslant C_{\ell,d} > 0, \tag{25}$$

which, together with (21), implies that

$$\psi \in C^\infty[0,\infty) \text{ and } \operatorname{supp}\psi \subset [0,\ 4\mu]. \tag{26}$$

Now invoking Lemma 4 three times, with $a = \eta$, ϕ and ψ, respectively, in view of (23), (26) and the fact that η is a C^∞-function with compact support, we obtain from (17) and (22) that for $1 \leqslant p \leqslant \infty$,

$$\sup_{t>0} \|V_t(f)\|_p + \sup_{t>0} \|\Phi_t(f)\|_p + \sup_{t>0} \|\Psi_t(f)\|_p \leqslant C_p \|f\|_p. \tag{27}$$

We claim that (18) and (19) follow from (27). In fact, from the identity

$$\left(f - V_{t/2\mu}(f)\right)^\wedge(\xi) = W(t\xi)(f - B_{\ell,t}(f))^\wedge(\xi),$$

where

$$W(\xi) := \left(1 - \eta(\frac{|\xi|}{2\mu})\right)\left(\frac{(m_\ell(|\xi|))^3}{1-m_\ell(|\xi|)} + 1 + m_\ell(|\xi|) + (m_\ell(|\xi|))^2\right),$$

it follows that

$$f - V_{t/2\mu}(f) = \Phi_t(f - B_{\ell,t}(f)) + (I - V_{t/2\mu})(I + B_{\ell,t} + B_{\ell,t}^2)(f - B_{\ell,t}(f)),$$

where I denotes the identity operator on $L^p(\mathbb{R}^d)$. This, together with (27) and the fact that $\|B_{\ell,t}\|_{(p,p)} \leqslant C_\ell$, gives (18).

Similarly, from the identities

$$\left(t^{2\ell}\triangle^\ell V_{t/2\mu}(f)\right)^\wedge(\xi) = \frac{(-1)^\ell t^{2\ell}|\xi|^{2\ell}\eta(\frac{t|\xi|}{2\mu})}{1 - m_\ell(t|\xi|)} (f - B_{\ell,t}(f))^\wedge(\xi)$$
$$= (-1)^\ell \Psi_t(f - B_{\ell,t}(f))^\wedge(\xi),$$

it follows that

$$t^{2\ell}\triangle^\ell V_{t/2\mu}(f) = (-1)^\ell \Psi_t(f - B_{\ell,t}(f)),$$

which, again by (27), implies (19). This completes the proof.

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